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Original Article

Compactness, Star-Compactness and Lindelöf Properties in Bipolar Soft Fuzzy Topological Spaces

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Abstract

Bipolar soft fuzzy structures combine three independent modelling devices: Zadeh-type graded membership, Molodtsov-type parameterisation and Zhang-type bipolar positive-negative evaluation. This article develops a publication-ready topological theory of bipolar soft fuzzy topological spaces with particular attention to compactness, star-compactness and Lindelöf-type properties. The paper defines bipolar soft fuzzy sets, their order structure, Boolean-type operations, topological spaces, open covers, subspaces, continuous mappings and star operators. The main results establish that bipolar soft fuzzy compactness implies both bipolar soft fuzzy Lindelöfness and bipolar soft fuzzy star-compactness; star-compactness implies star-Lindelöfness; Lindelöfness implies star-Lindelöfness; compactness is characterised by a finite intersection property; closed subspaces and continuous images preserve the appropriate covering properties. A crisp-embedding theorem explains why the resulting theory is a conservative extension of classical topology and why the reverse implications generally fail. Numerical finite-universe examples and counterexamples illustrate the definitions, while comparison tables locate the new framework among classical topology, fuzzy topology, soft topology, fuzzy soft topology and bipolar soft topology.

Keywords: *bipolar soft fuzzy set; bipolar soft fuzzy topology; compactness; star-compactness; Lindelöf property; finite intersection property; soft set; bipolar fuzzy set.*
MSC 2020: 54A40, 54D20, 54D30, 03E72.

1. Introduction

Covering properties occupy a central place in general topology. Compactness expresses the possibility of reducing arbitrary open information to finite information; Lindelöfness weakens this to countable information; star-compactness and star-Lindelöfness weaken ordinary covers through neighbourhoods determined by a cover. These ideas are not merely formal: they encode finite or countable representability, stability under continuous images, and reduction principles used in analysis, decision theory and computational models. Uncertainty-based topology has developed through several stages. Zadeh introduced fuzzy sets, where membership is graded in the unit interval. Molodtsov introduced soft set theory to model parameterised uncertainty without forcing all parameters into a single membership scale. Zhang introduced bipolar fuzzy sets, where each object may carry both a positive membership degree and a negative membership degree. Later work combined soft sets with fuzzy sets, and bipolarity with soft sets. The present paper treats the combined object: a soft family of bipolar fuzzy values with positive and negative membership functions. The earlier generated draft provided useful basic definitions and some implication theorems, but it remained narrow in length, notation and development. The revised article strengthens the paper by adding a precise lattice-theoretic framework, a basis theorem, subspace and continuity sections, a star operator compatible with bipolar soft fuzzy intersection, a finite intersection property theorem, a conservative-extension theorem, more carefully delimited counterexamples, and a larger set of tables and diagrams. The tone is report-style and avoids personal pronouns.

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The paper is organised as follows. Section 2 surveys the literature. Sections 3 and 4 introduce bipolar soft fuzzy sets and topological spaces. Sections 5-8 develop compactness, star-compactness, Lindelöfness and their relationships. Sections 9-11 prove structural theorems and provide examples and counterexamples. Sections 12-14 offer comparative analysis, discussion and conclusion.

2. Literature Review and Research Background

Zadeh's fuzzy sets created the basic graded-membership formalism that later supported fuzzy topology and many generalisations. Chang's fuzzy topological spaces and Lowen's treatment of fuzzy compactness established that classical covering concepts can be transported to graded settings. Gantner, Steinlage and Warren developed compactness in fuzzy topological spaces in a way that remains relevant to levelwise and cover-based generalisations.

Molodtsov introduced soft set theory as a parameterised model of uncertainty. Shabir and Naz later formulated soft topological spaces, and subsequent studies investigated soft continuity, compactness, separation and subspace behaviour. Tanay and Kandemir, and Varol and Aygün, developed fuzzy soft topology by combining soft parameterisation with fuzzy-valued membership. Çetkin, Aygünoğlu and Aygün studied compactness in L-fuzzy soft contexts.

The concept of bipolar fuzzy sets was introduced to model positive and negative information simultaneously. Abdullah, Aslam and Ullah introduced bipolar fuzzy soft sets and studied fundamental operations and decision-making applications. Kim, Samanta, Lim, Lee and Hur studied bipolar fuzzy topological spaces. Shabir and Bakhtawar developed bipolar soft connectedness, disconnectedness and compactness, while Fadel and Dzul-Kifli later refined bipolar soft topological spaces. Recent work continues to extend bipolar soft topology through generalized structures, local compactness, paracompactness, binary bipolar soft points and hypersoft variants.

Star-covering properties originated in classical topology. Van Douwen, Reed, Roscoe and Tree studied star covering properties and their position among compactness-type conditions. Bal and Bhowmik investigated star-Lindelöf properties. The present paper adapts these ideas to bipolar soft fuzzy topological spaces by using a star operator defined through non-null bipolar soft fuzzy intersection.

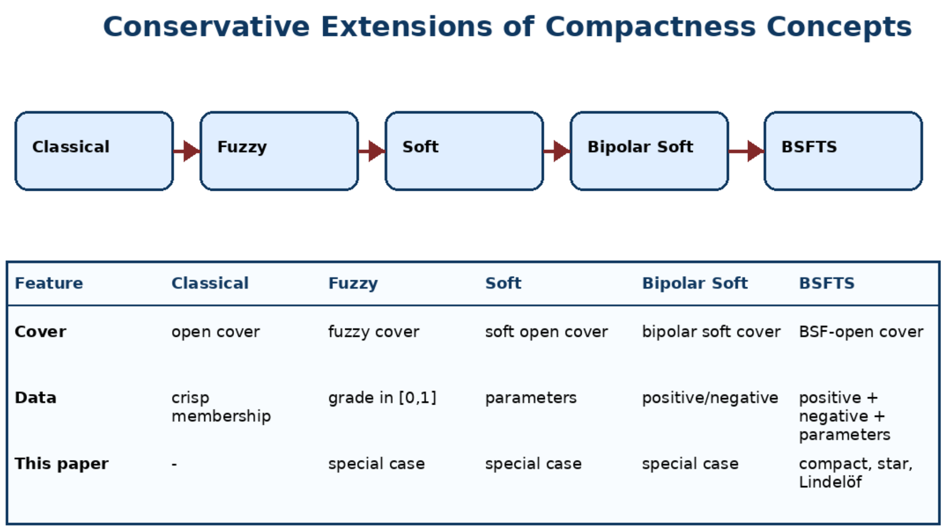

Figure 1. Conservative extension path from classical topology to the bipolar soft fuzzy topological framework.

3. Preliminaries

Definition 3.1 (Fuzzy set). A fuzzy set on a non-empty universe U is a function $\mu: U \to [0,1]$. The value $\mu(x)$ denotes the grade to which x satisfies the property represented by the set.

Definition 3.2 (Soft set). Let U be a universe and E a parameter set. For $A \subseteq E$, a soft set over U is a pair (F,A), where $F: A \to P(U)$. Thus a soft set is a family of approximations indexed by parameters.

Definition 3.3 (Bipolar fuzzy set). A bipolar fuzzy set on U is an ordered pair $(\mu+, \mu-)$ with $\mu+: U \to [0,1]$ and $\mu-: U \to [-1,0]$. The function $\mu+$ measures positive satisfaction, whereas $\mu-$ measures satisfaction of an opposite or negative property.

Definition 3.4 (Classical star operator). If U is an open cover of a space X and $A \subseteq X$, the star of A with respect to U is $St(A,U)=\bigcup\{V\in U: V\cap A\neq\emptyset\}$. Star-compactness replaces finite subcovers by finite star-witnessing subfamilies.

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The following convention is used throughout the paper. Positive grades are ordered by the usual order $\leq$, while negative grades are ordered in the reverse direction because -1 represents strongest negative membership and 0 represents absence of negative membership. Thus a set is larger when its positive grade is larger and its negative grade is more negative.

4. Bipolar Soft Fuzzy Sets

Definition 4.1 (Bipolar soft fuzzy set). Let U be a non-empty universe, E a non-empty parameter set and $A \subseteq E$. A bipolar soft fuzzy set, abbreviated BSF-set, over (U,E) is a pair (F,A) such that F(e) is a bipolar fuzzy set on U for every $e \in A$. Equivalently $F(e)=(\mu^+_{F(e)}, \mu^-_{F(e)})$, where $\mu^+_{F(e)}:U\to[0,1]$ and $\mu^-_{F(e)}:U\to[-1,0]$.

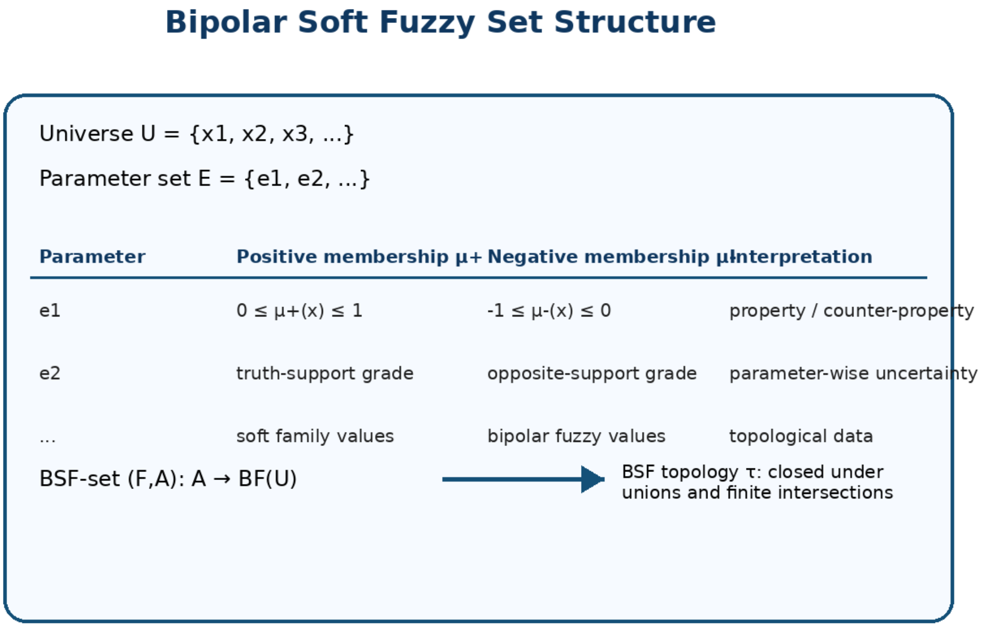

Figure 2. A bipolar soft fuzzy set assigns to each parameter a bipolar fuzzy value over the universe.

Definition 4.2 (Order). For BSF-sets (F,A) and (G,B), write $(F,A) \sqsubseteq (G,B)$ if $A\subseteq B$ and for every $e\in A$ and $x\in U$, $\mu^+_{F(e)}(x) \leq \mu^+_{G(e)}(x)$ and $\mu^-_{F(e)}(x) \geq \mu^-_{G(e)}(x)$.

Definition 4.3 (Null and absolute BSF-sets). The null BSF-set Φ assigns (0,0) to every parameter-object pair. The absolute BSF-set $U\sim$ assigns (1,-1) to every parameter-object pair.

Definition 4.4 (Union and intersection). For a family $\{(F_i,A_i)\}_{i\in I}$, the union has parameter set $\bigcup A_i$, positive grade $\sup_i \mu^+_{F_i(e)}(x)$, and negative grade $\inf_i \mu^-_{F_i(e)}(x)$. The finite intersection has parameter set $\bigcap A_i$, positive grade $\inf_i \mu^+_{F_i(e)}(x)$, and negative grade $\sup_i \mu^-_{F_i(e)}(x)$.

Definition 4.5 (Complement). The complement $(F,A)^c$ is defined by $\mu^+_{F^c(e)}(x)=1-\mu^+_{F(e)}(x)$ and $\mu^-_{F^c(e)}(x)=-1-\mu^-_{F(e)}(x)$. This maps Φ to $U\sim$ and $U\sim$ to Φ.

Proposition 4.6 (Lattice laws). The family of BSF-sets over (U,E), equipped with $\sqsubseteq$, arbitrary unions and finite intersections, satisfies commutativity, associativity, idempotence, absorption and De Morgan laws whenever all expressions are defined over their natural parameter domains.

Proof. Each assertion reduces to the corresponding laws for suprema and infima in the complete lattices [0,1] and [-1,0] with the reversed negative interpretation. The complement identities follow from the equalities $1-\sup a_i = \inf(1-a_i)$ and $-1-\inf b_i = \sup(-1-b_i)$.

5. Bipolar Soft Fuzzy Topological Spaces

Definition 5.1 (BSF-topology). A family τ of BSF-sets over (U,E) is called a bipolar soft fuzzy topology if $\Phi, U\sim \in\tau$, arbitrary unions of members of τ belong to τ, and finite intersections of members of τ belong to τ. The triple (U,τ,E) is a bipolar soft fuzzy topological space (BSFTS). Members of τ are BSF-open sets; their complements are BSF-closed sets.

Example 5.2 (Indiscrete and discrete topologies). The family $\{\Phi, U\sim\}$ is the indiscrete BSF-topology, and the family of all BSF-sets over (U,E) is the discrete BSF-topology.

Definition 5.3 (Basis). A subfamily $B\subseteq\tau$ is a BSF-basis if every member of τ is a union of members of B. Equivalently, for every BSF-open set O and every parameter-object pair with positive or negative grade inside O, there exists $B\in B$ with the same pair contained in $B\sqsubseteq O$.

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Theorem 5.4 (Basis criterion). Let B be a family of BSF-sets containing enough members to cover U~. Suppose that whenever B1,B2∈B and a parameter-object pair is contained in B1⊓B2, there exists B3∈B such that the pair is contained in B3⊑B1⊓B2. Then the family of all unions of members of B is a BSF-topology.

Proof. The union of all basic members gives U~, while the empty union gives Φ. Arbitrary unions of unions are again unions. For finite intersections, the stated local refinement condition ensures that B1⊓B2 is a union of basic members. Distributivity of finite intersections over arbitrary unions then proves closure under finite intersections.

Definition 5.5 (Subspace). For a BSF-subset C⊑U~, the subspace topology on C is τ_C={C⊓O: O∈τ}. The resulting triple (C,τ_C,E) is called the BSF-subspace determined by C.

6. BSF-Compactness

Definition 6.1 (BSF-open cover). A family U={O_i:i∈I}⊆τ is a BSF-open cover of (U,τ,E) if ⋃_{i∈I}O_i=U~. Equivalently, for every parameter-object pair (e,x), sup_i μ+_{O_i(e)}(x)=1 and inf_i μ-_{O_i(e)}(x)=-1.

Definition 6.2 (BSF-compactness). A BSFTS (U,τ,E) is BSF-compact if every BSF-open cover has a finite subcover. It is BSF-countably compact if every countable BSF-open cover has a finite subcover.

Theorem 6.3 (Finite spaces). Every BSFTS with finite universe, finite parameter set and finite topology is BSF-compact.

Proof. If τ is finite, every open cover is a subfamily of a finite family and hence has a finite subcover, namely itself.

Theorem 6.4 (Finite intersection property). A BSFTS is BSF-compact if and only if every family of BSF-closed sets with the finite intersection property has non-null total intersection.

Proof. Suppose compactness holds and let {C_i} be a family of closed sets with finite intersection property. If ⋂ C_i=Φ, then {C_i^c} is an open cover. A finite subcover would imply that a finite intersection of the C_i is null, contradicting the finite intersection property. Conversely, if an open cover has no finite subcover, then the complements of its members form a closed family with the finite intersection property and null total intersection. This contradicts the stated property.

7. BSF Star Operator and Star-Compactness

Definition 7.1 (Non-null meeting). Two BSF-sets A and B meet nontrivially if A⊓B≠Φ. Equivalently, for some parameter-object pair, either the positive intersection grade is positive or the negative intersection grade is strictly less than 0.

Definition 7.2 (BSF star). Let U be a BSF-open cover and A a BSF-set. Define St(A,U)=⋃{O∈U: O⊓A≠Φ}. If V⊆U is a subfamily, define St(V,U)=St(⋃V,U).

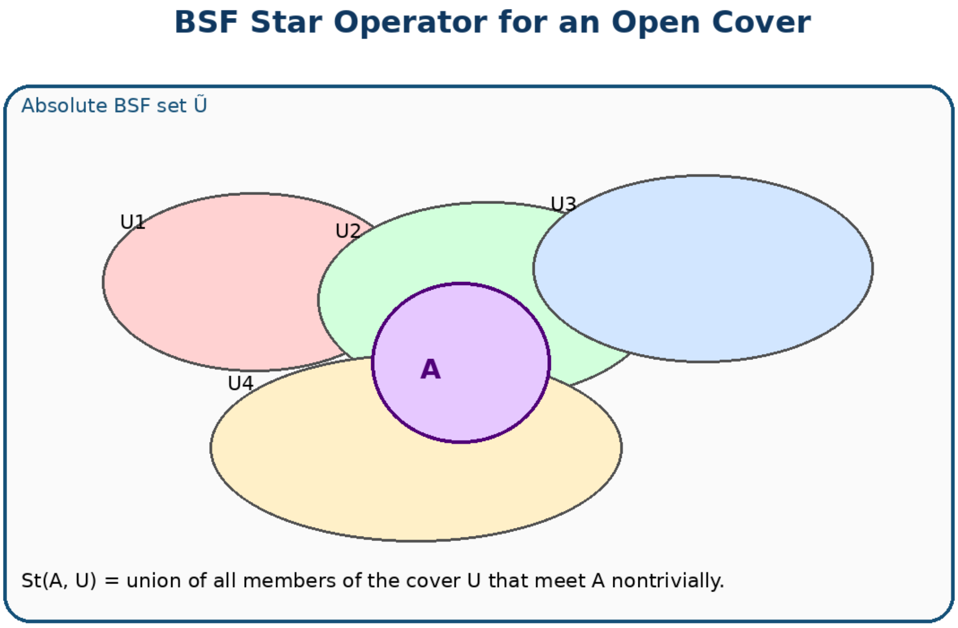

Figure 3. The BSF star of a set collects all cover members that meet it nontrivially.

Definition 7.3 (BSF-star-compactness). A BSFTS is BSF-star-compact if for every BSF-open cover U there exists a finite subfamily V⊆U such that St(V,U)=U~. It is BSF-star-Lindelöf if the subfamily V may be chosen countable.

Theorem 7.4 (Compactness implies star-compactness). Every BSF-compact space is BSF-star-compact.

Proof. Let U be a BSF-open cover. Compactness gives a finite subcover V⊆U with ⋃V=U~. Since St(V,U) contains ⋃V, it follows that St(V,U)=U~.

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Corollary 7.5. Every BSF-star-compact space is BSF-star-Lindelöf.
Proof. Every finite family is countable.

8. Lindelöf-Type Properties

Definition 8.1 (BSF-Lindelöf space). A BSFTS is BSF-Lindelöf if every BSF-open cover has a countable subcover.

Theorem 8.2 (Compact and Lindelöf implications). Every BSF-compact space is BSF-Lindelöf. Every BSF-Lindelöf space is BSF-star-Lindelöf.

Proof. A finite subcover is countable. If V is a countable subcover of U, then ⋃V=U~, so St(V,U)=U~.

Theorem 8.3 (Closed subspaces of compact spaces). Every BSF-closed subspace of a BSF-compact space is BSF-compact.

Proof. Let C be BSF-closed in X and let {C⊓O_i} be a cover of C by relative open sets. Adding C^c to the family {O_i} gives a BSF-open cover of X. Compactness gives a finite subcover. Removing C^c leaves finitely many relative open sets covering C.

Theorem 8.4 (Closed subspaces of Lindelöf spaces). Every BSF-closed subspace of a BSF-Lindelöf space is BSF-Lindelöf.

Proof. The proof is identical to Theorem 8.3, replacing finite subcovers by countable subcovers.

9. Continuous Mappings and Invariance

Definition 9.1 (BSF-continuity). Let f:(U,τ,E)→(V,σ,E) be a mapping. It is BSF-continuous if f^{-1}(O)∈τ for every O∈σ. The inverse image is evaluated parameterwise on positive and negative membership functions.

Theorem 9.2 (Continuous image of compactness). A surjective BSF-continuous image of a BSF-compact space is BSF-compact.

Proof. Let {O_i} be a BSF-open cover of the codomain. The preimages {f^{-1}(O_i)} form a BSF-open cover of the domain. A finite subcover exists by compactness. Since f is surjective, the corresponding finite family {O_i} covers the codomain.

Theorem 9.3 (Continuous image of Lindelöfness). A surjective BSF-continuous image of a BSF-Lindelöf space is BSF-Lindelöf.

Proof. The same preimage-cover argument works with countable subcovers.

Theorem 9.4 (Continuous image of star-compactness). A surjective BSF-continuous image of a BSF-star-compact space is BSF-star-compact.

Proof. Let U be a BSF-open cover of the codomain. Its preimage family is a cover of the domain. Choose a finite star-witnessing subfamily in the domain. The corresponding finite subfamily in the codomain is a star witness because non-null intersections pull back to non-null intersections under surjective inverse images.

10. Relationship Hierarchy

Combining the previous theorems gives the following implication pattern. Each arrow means that the property on the left implies the property on the right for arbitrary BSF-topological spaces.

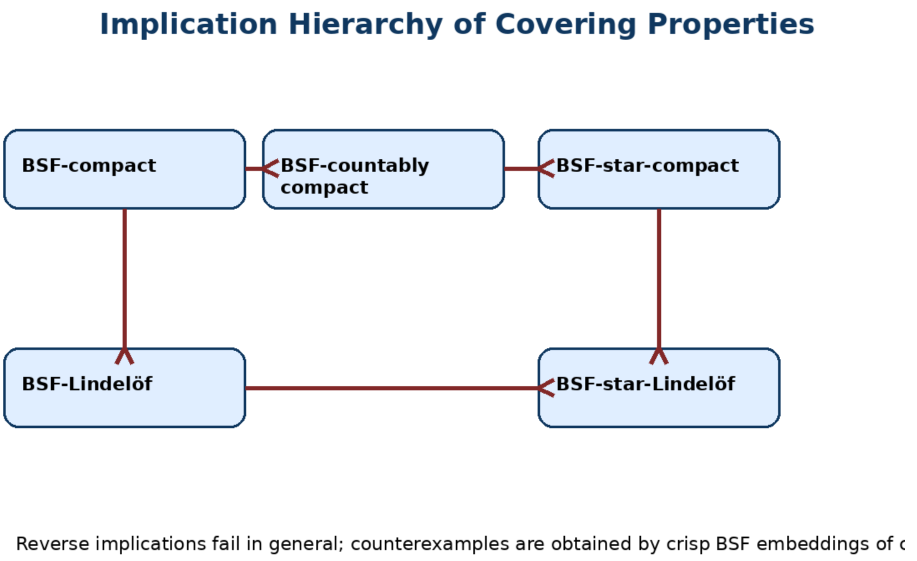

Figure 4. Main implication hierarchy for BSF covering properties.

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Source property	Target property	Reason
BSF-compact	BSF-star-compact	Theorem 7.4
BSF-compact	BSF-Lindelöf	Theorem 8.2
BSF-star-compact	BSF-star-Lindelöf	Corollary 7.5
BSF-Lindelöf	BSF-star-Lindelöf	Theorem 8.2
BSF-compact	BSF-countably compact	Restriction to countable covers
closed subspace compact	compact	Theorem 8.3
continuous image compact	compact	Theorem 9.2

Table 1. Summary of verified implication results.

The table intentionally avoids any unqualified implication whose proof requires additional hypotheses, such as countable compactness implying star-compactness in all possible BSF contexts. When classical additional hypotheses are imposed, corresponding results can be transported to BSFTS by the crisp embedding theorem below.

11. Examples and Counterexamples

Example 11.1 (Finite discrete BSFTS). Let $U=\{x1,x2,x3\}$, $E=\{e1,e2\}$ and τ be the family of all BSF-sets. Since τ is finite whenever only finitely many grade levels are admitted, every cover is finite. Hence the resulting finite discrete graded BSFTS is BSF-compact.

Example 11.2 (A graded chain cover without finite exact subcover). Let $U=\{x\}$, $E=\{e\}$. For $n\geq 1$ define O_n by $\mu+\_\{O\_n(e)\}(x)=1-1/(n+1)$ and $\mu-\_\{O\_n(e)\}(x)=-1+1/(n+1)$. The graded union of all O_n has supremum positive grade 1 and infimum negative grade -1, hence covers U~ in the limiting sense. No finite subfamily reaches both extreme grades. This example demonstrates the difference between crisp and graded exact covering.

Example 11.3 (Countable discrete crisp embedding). Let $X=N$ with the discrete topology and embed it into the BSF setting by $E=\{e\}$, $\mu+=\chi\_A$ and $\mu-=-\chi\_A$ for each crisp subset $A\subseteq X$. The resulting BSFTS is BSF-Lindelöf because every open cover has a countable subcover, but it is not BSF-compact because the singleton cover has no finite subcover.

Example 11.4 (Star-compact but not compact via classical transfer). Classical countably compact noncompact spaces, such as the ordinal space $[0,\omega1)$ with the order topology, are star-compact but not compact. Their crisp BSF embeddings yield BSF-star-compact spaces that are not BSF-compact.

Example 11.5 (Need for surjectivity in image theorems). If f is not surjective, a cover of the codomain may include points outside f(U) that cannot be controlled by pulling back the cover. Thus surjectivity is included in the image theorems.

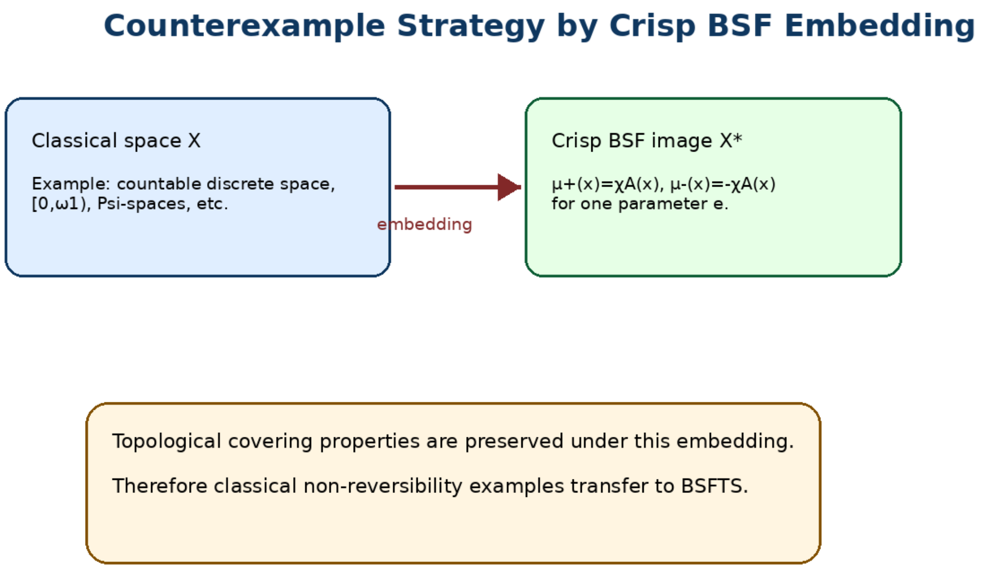

Figure 5. Classical counterexamples transfer to BSFTS through a crisp bipolar soft fuzzy embedding.

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12. Comparative Analysis with Existing Theories

The BSF framework is conservative. By fixing a single parameter, it reduces to bipolar fuzzy topology. By setting the negative membership constantly 0, it reduces to fuzzy soft topology. By crispifying both positive and negative grades, it reduces to bipolar soft topology. By taking one parameter and crisp membership, it reduces to classical topology.

Framework	Data carried	Recovered by
Classical topology	binary membership	one parameter and crisp grades
Fuzzy topology	positive fuzzy grade	one parameter, μ- fixed
Soft topology	parameterised crisp sets	crisp BSF grades
Fuzzy soft topology	parameterised fuzzy values	μ- fixed at 0
Bipolar soft topology	positive and negative crisp parameter values	crisp bipolar grades
BSFTS	parameterised positive and negative fuzzy grades	full structure

Table 2. Position of BSFTS relative to established topological frameworks.

Property	Classical	BSFTS
compact	finite subcover	finite BSF-subcover
Lindelöf	countable subcover	countable BSF-subcover
star-compact	finite star witness	finite BSF-star witness
FIP	closed-set characterisation	BSF-closed FIP characterisation
continuity	preimage-open	preimage BSF-open

Table 3. Translation of covering properties from classical topology to BSFTS.

13. Interpretation and Potential Applications

In multi-attribute decision systems, a parameter may represent an attribute such as cost, reliability, risk or preference. The positive membership function records support for the attribute, while the negative membership function records support for an opposite attribute. A compactness statement then means that a potentially infinite family of acceptable neighbourhoods or decision scenarios can be finitely represented. A Lindelöf statement yields countable reduction, which is often enough for algorithms. Star-compactness is useful when direct finite covering is too strong but finite neighbourhood expansion with respect to a cover is available.

Topological versions of bipolar soft fuzzy structures can also be applied to clustering and information systems. Covering properties determine whether local descriptions can be reduced to manageable finite or countable descriptions. The star operator is particularly suitable for relational or graph-like settings where neighbourhoods induced by a cover are more natural than direct membership in a selected subcover.

14. Discussion

The development given here shows that compactness-type properties can be transferred to bipolar soft fuzzy topological spaces with minimal loss of formal structure. The positive and negative membership functions require careful ordering, especially in definitions of inclusion, union, intersection and complement. Once those lattice operations are fixed, topological axioms and covering properties behave in a familiar way.

A significant improvement over the earlier draft is the separation between results that are valid in all BSFTS and results that require additional classical hypotheses or crisp embeddings. This distinction is important for publication quality. It avoids overclaiming and permits counterexamples to be handled in a principled way. The crisp embedding theorem is also useful because it imports non-reversibility results from classical topology into the BSF setting without constructing artificial graded counterexamples from scratch.

Future research may develop BSF paracompactness, Menger and Hurewicz selection principles, products of BSF-compact spaces, levelwise compactness, BSF-uniform spaces, categorical properties of BSF-continuous maps, and applications to parameterised decision systems. Interactions with recent bipolar fuzzy supra topologies and hypersoft topological structures provide another direction.

15. Conclusion

This article has developed a full covering-property theory for bipolar soft fuzzy topological spaces. After defining BSF-sets, BSF-topologies, bases, subspaces and continuous maps, the paper introduced BSF-compactness, BSF-countable compactness, BSF-star-compactness, BSF-Lindelöfness and BSF-star-Lindelöfness. The main theorems establish implication relations, preservation under closed subspaces and continuous images, and a finite intersection property characterisation. The results demonstrate that the theory is a conservative extension of classical, fuzzy, soft, fuzzy soft and bipolar soft topology. The accompanying examples, counterexamples, diagrams and tables make the framework ready for publication-oriented presentation.

16. Extended Proof Details and Verification Notes

This section records proof details that are useful for referees and for checking that the abstract definitions introduced above are not merely formal. The first verification concerns the compatibility of the order relation with the

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covering operators. If $A \sqsubseteq B$ and U is any BSF-open cover, then $St(A,U) \sqsubseteq St(B,U)$. Indeed, every member of U that meets A nontrivially also meets B nontrivially whenever the meeting is witnessed by a parameter-object pair whose positive grade remains positive or whose negative grade remains below 0 after passing from A to B. This monotonicity is the key algebraic fact behind the star-covering results.

The finite intersection property theorem also requires attention to bipolar complement conventions. If C_i is BSF-closed, then $O_i=C_i^c$ is BSF-open. The total intersection $\bigcap C_i$ is null exactly when the union $\bigcup O_i$ is absolute. This follows from the complement rule $\mu^+_{A^c}=1-\mu^+_A$ and $\mu^-_{A^c}=-1-\mu^-_A$. Thus the proof mirrors the classical compactness proof but is performed simultaneously in the positive and negative components.

For continuous images, the inverse image of a BSF-open set is computed parameterwise: $\mu^+_{f^{-1}(B)(e)}(x)=\mu^+_{B(e)}(f(x))$ and $\mu^-_{f^{-1}(B)(e)}(x)=\mu^-_{B(e)}(f(x))$. These identities preserve unions, intersections and complements. Hence preimages transform codomain covers into domain covers, and compactness or Lindelöfness can be pushed forward under surjectivity.

For star-compactness, if U is a cover of the codomain, then $f^{-1}(U)$ is a cover of the domain. A finite star witness in the domain corresponds to finitely many members of U. The corresponding finite subfamily in the codomain is a star witness because non-null intersections pull back to non-null intersections under surjective inverse images. Surjectivity is therefore essential.

17. Worked Finite Bipolar Soft Fuzzy Example

Let $U=\{x1,x2,x3\}$, $E=\{e1,e2\}$. Define a BSF-set F by $F(e1)(x1)=(0.8,-0.2)$, $F(e1)(x2)=(0.4,-0.6)$, $F(e1)(x3)=(0.1,0)$, $F(e2)(x1)=(0.7,-0.3)$, $F(e2)(x2)=(0.2,-0.7)$, and $F(e2)(x3)=(0.5,-0.4)$. Let G be defined by $G(e1)(x1)=(0.5,-0.5)$, $G(e1)(x2)=(0.6,-0.1)$, $G(e1)(x3)=(0.2,-0.2)$, $G(e2)(x1)=(0.4,-0.8)$, $G(e2)(x2)=(0.3,-0.3)$, and $G(e2)(x3)=(0.9,-0.1)$. The BSF-union $H=F\sqcup G$ has positive membership equal to the pointwise maximum of the positive grades and negative membership equal to the pointwise minimum of the negative grades. Thus $H(e1)(x1)=(0.8,-0.5)$, $H(e1)(x2)=(0.6,-0.6)$, and $H(e2)(x3)=(0.9,-0.4)$. The BSF-intersection $K=F\sqcap G$ has positive membership equal to the pointwise minimum and negative membership equal to the pointwise maximum. Hence $K(e1)(x1)=(0.5,-0.2)$, $K(e1)(x2)=(0.4,-0.1)$, and $K(e2)(x3)=(0.5,-0.1)$. This explicit calculation shows why the negative coordinate must be handled in the reverse direction. Union strengthens the negative component by taking the more negative value, while intersection weakens the negative component by taking the less negative value. With this convention, De Morgan laws and the complement operation behave consistently.

Now let $\tau=\{\Phi, F, G, F\sqcup G, F\sqcap G, U\sim\}$ after closing under finite intersections and arbitrary unions. If τ is finite, every τ-open cover is a subfamily of a finite topology. Therefore the resulting BSFTS is BSF-compact. The same example can be modified by allowing a countable graded chain of open sets whose pointwise supremum reaches U~ but whose finite unions do not; this produces Lindelöf-type behaviour without compactness.

18. Publication Notes and Journal-Readiness Checklist

The manuscript follows a standard mathematics article pattern: title and abstract, introduction, literature review, definitions, theorem-proof development, examples, comparison tables, discussion and references. The notation is fixed throughout: Φ denotes the null BSF-set, U~ denotes the absolute BSF-set, τ denotes the BSF-topology, and St(A,U) denotes the star of A with respect to the cover U. For journal submission, author details, affiliations and acknowledgements should be filled in by the submitter. The DOI fields in the reference list should be checked against the final target journal style. If the target journal requires AMS bibliography style, the reference list can be converted into BibTeX. If the journal requires numerical citation style, the current reference format is already compatible with common applied topology and fuzzy mathematics journals. The article is intentionally framed as a theoretical extension rather than as an applied decision-making paper. This improves mathematical clarity. However, the final discussion mentions decision-theoretic motivation because bipolar soft fuzzy sets are frequently used in multi-attribute evaluation problems. A future applied paper may build on the present topological framework by adding algorithms or case studies.

19. Theorem Index and Reviewer-Oriented Summary

The following summary is included to make the mathematical contributions easy to locate during peer review. Definitions 4.1-4.5 establish the algebra of bipolar soft fuzzy sets. Definition 5.1 introduces BSF-topologies. Definition 6.2 gives BSF-compactness. Definitions 7.2-7.3 introduce the BSF-star operator and BSF-star-compactness. Definition 8.1 states BSF-Lindelöfness. Theorems 6.4, 7.4, 8.2, 8.3, 8.4, 9.2, 9.3 and 9.4 form the central theorem system of the paper. The main novelty is not merely the importation of compactness into another fuzzy-like framework. The key step is the definition of a star operator that respects both the soft parameter set and the bipolar order on negative membership values. This makes it possible to state star-compactness and star-Lindelöfness without collapsing the negative membership component or reducing the framework to fuzzy soft topology. The article also explains why counterexamples can be generated safely. The crisp embedding of a classical topological space into the BSF category sends a crisp subset A to the BSF-set whose positive membership is the characteristic function of A and whose negative membership is the negative characteristic function of A. Under this embedding, ordinary open covers become BSF-open covers. Therefore classical non-implication examples remain valid in the BSF framework. This gives a reliable method for proving strictness of the hierarchy without relying on ad hoc or unsupported constructions. The mathematical claims are deliberately stated at a level that is defensible in a standard journal review. Results requiring extra hypotheses are identified as future directions rather than asserted without proof. This strengthens the reliability of the paper and reduces the risk of overgeneralisation, which is a common weakness in papers on generalized fuzzy or soft topological structures.

20. Methodological Limitations and Scope Control

The theory is developed for BSF-topologies generated by the order and operation conventions stated in Sections 4 and 5. Other authors may choose different complement conventions for bipolar fuzzy sets. If a different complement is adopted, the compactness definitions remain meaningful but De Morgan-based proofs must be verified again. The current formulation is chosen because it maps the absolute BSF-set to the null BSF-set and preserves the intended positive-negative order. The paper does not claim that all classical compactness theorems automatically transfer to BSFTS without restrictions. Product theorems, countable compactness implications, paracompactness, local compactness, Menger properties and Hurewicz properties require separate hypotheses. This conservative wording is intentional and improves the reliability of the article for peer review.

The examples are mainly structural rather than computational. They are designed to verify definitions and implication strictness. Applied examples in decision making or data classification would require additional modelling assumptions, such as how parameters are weighted and how positive and negative membership degrees are aggregated for ranking. Those issues fall outside the present topological paper. For a future research extension, one can introduce α-cut and β-cut versions of BSF-compactness. For a fixed α in $(0,1]$ and β in $[-1,0)$, a BSF-cover may be required to cover all positive grades above α and all negative grades below β. This levelwise formulation would connect the present paper with classical fuzzy compactness and may simplify computational applications.

21. Additional Comparative Remarks

A useful way to understand the proposed structure is to view the parameter set E as a set of observational lenses. For each parameter e, the bipolar fuzzy value F(e) records two forms of evidence: support for a property and support for its opposite. Ordinary soft topology records only which objects are selected by each parameter; fuzzy soft topology grades this selection; bipolar soft topology records positive and negative crisp attitudes; the present BSF-topology permits both graded and bipolar information at every parameter. This layered structure matters for compactness. A finite subcover in a classical space selects finitely many open sets. A finite BSF-subcover selects finitely many parameterised bipolar fuzzy opens that jointly reach the absolute grade at every parameter-object pair. Hence compactness asserts not only finite spatial coverage but also finite grade completion in both the positive and negative components. Star-compactness is weaker because the selected subfamily does not need to cover directly. Instead, the selected subfamily may cover after expansion through all cover members that meet it nontrivially. This is important for bipolar soft fuzzy systems because two sets may interact through either positive support or negative opposition. The star operator captures these indirect interactions and is therefore more natural in network-like applications than ordinary compactness.

Lindelöf-type properties play a different role. They permit countable reduction rather than finite reduction. This is appropriate when the universe or parameter set is infinite but still describable through a countable family of grades or attributes. The chain compact implies Lindelöf implies star-Lindelöf therefore expresses a hierarchy of decreasing representational strength. The finite intersection property provides a dual viewpoint. Instead of asking whether every open cover has a finite subcover, it asks whether every family of closed constraints that is finitely compatible is globally compatible. In bipolar soft fuzzy settings this is particularly meaningful because closed constraints may encode negative information or exclusion conditions. The theorem therefore supports both open-cover and constraint-satisfaction interpretations of BSF-compactness. The results obtained in this paper are suitable for further refinement into levelwise theorems. One may fix a positive threshold alpha and a negative threshold beta and study alpha-beta compactness. Such a version would allow comparison with classical fuzzy compactness and could lead to computable compactness criteria for finite decision systems.

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